1D Kinematics

Last Updated: June 5, 2018

"Kinematics" is the study of the motion of particles and objects (though not the *causes* of that motion, which is the subject of Dynamics). This is the realm of position, displacement, velocity, speed, and acceleration; we should begin by defining these terms:

Definition: Basic Kinematics Terms

- **Position**: An identification of the location of a particle in some coordinate space. Often we'll denote a position at time t as a vector quantiy $\mathbf{r}(t)$, though in one dimension, this reduces to a scalar.
- **Displacement**: The distance between two positions, often used to denote how far a particle has traveled between two times: $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0$. Often we'll assume a particle starts at the origin, so $\Delta\mathbf{r} \to \mathbf{r}$.
- **Velocity**: The instantaneous change in position:
$$\mathbf{v} \equiv \frac{d\mathbf{r}}{dt} \tag{1}$$
- **Speed**: The magnitude of velocity:
$$s \equiv |\mathbf{v}| \tag{2}$$
- **Acceleration**: The instantaneous change in velocity:
$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{3}$$

One Dimension

Right now we're looking at one-dimensional kinematics. This means that all motion is either postive or negative (no sideways). Think of moving up and down in an elevator or how cars on a dragstrip just move straight forward. In this case, the direction of a vector can be reduced to a simple sign: positive for forward or up, and negative for backwards or down.

Constant Acceleration

A common scenario for motion in one dimension is that of constant acceleration. For example, an object dropped a short distance undergoes constant acceleration due to gravity. In this specialized case, we can derive a series of four useful equations that relate positions/displacements, velocities, and acceleration. Let's derive them now.

Deriving the Kinematic Equations

Since we are dealing with only one dimension, we'll defer on vector notation and simply rely on the sign of scalars to indicate direction. We start with the fact that acceleration is constant:

$$\frac{da}{dt} = 0 \tag{4}$$

From the definition of velocity, (1), we can surmise that the velocity is some linear function of time, since linear functions are the only functions whose derivatives are a constant:

$$\frac{dv}{dt} = a \tag{5}$$
$$v(t) = at + (\text{some constant}) \tag{6}$$

We don't know what the constant term in (6) is, but we know that it must be the velocity at $t = 0$ since it is the only non-vanishing term when $t = 0$. So, let's just call it v_0: the velocity at $t = 0$. This is the first kinematic equation:

$$\boxed{v(t) = v_0 + at} \tag{7}$$

We can take this a step further to find the position as a function of time starting with (1):

$$\frac{dr}{dt} = v \tag{8}$$
$$\frac{dr}{dt} = v_0 + at \tag{9}$$
$$r(t) = v_0 t + \frac{1}{2}at^2 + \text{some constant} \tag{10}$$

where we have again taken an anti-derivative of (9) to arrive at (10). Again we see that at $t = 0$, the only term remaining is the arbitrary constant, so we call this r_0 to get our second kinematic equation:

$$\boxed{r(t) = r_0 + v_0 t + \frac{1}{2}at^2} \tag{11}$$

These two equations are quite powerful in allowing us to start from a known state (r_0 and v_0) to some unknown future state ($r(t)$ and $v(t)$) at some future time t. What if we don't know how much time has elapsed, but we *do* know something about the end state? Then we would want to eliminate t from our equations. Let's solve (7) for t:

$$v(t) = v_0 + at \tag{12}$$
$$t = \frac{v(t) - v_0}{a}, \tag{13}$$

and plug that result in to (11) to see what we can come up with:

$$r(t) = r_0 + v_0 t + \frac{1}{2}at^2 \tag{14}$$
$$r(t) = r_0 + v_0\left[\frac{v(t) - v_0}{a}\right] + \frac{1}{2}a\left[\frac{v(t) - v_0}{a}\right]^2 \tag{15}$$
$$r(t) - r_0 = \frac{1}{2a}\left\{2v_0[v(t) - v_0] + [v(t) - v_0]^2\right\} \tag{16}$$
$$2a\left[r(t) - r_0\right] = 2v_0 v(t) - 2v_0^2 + v(t)^2 - 2v(t)v_0 + v_0^2 \tag{17}$$

After some canceling, we arrive at the third kinematic equation:

$$\boxed{v(t)^2 - v_0^2 = 2a\left[r(t) - r_0\right]} \tag{18}$$

We now have equations that allow us to ignore position/displacement (7), final velocity (11), and time (18). The last variable to eliminate is acceleration itself. If we re-arrange (7) again to solve for acceleration in terms of initial and final velocity, we get

$$a = \frac{v - v_0}{t} \tag{19}$$

Plugging this value for acceleration into (18), we get

$$v(t)^2 - v_0^2 = 2\left[\frac{v(t) - v_0}{t}\right][r(t) - r_0] \tag{20}$$

$$[v(t) - v_0][v(t) + v_0] = \frac{2}{t}[v(t) - v_0][r(t) - r_0] \tag{21}$$

$$v(t) + v_0 = \frac{2}{t}[r(t) - r_0] \tag{22}$$

where we've used the difference of squares identity $a^2 - b^2 = (a+b)(a-b)$ to get (21). Rearranging this recovers ths usual form of the last kinematic equation:

$$\boxed{r(t) - r_0 = \frac{v(t) + v_0}{2}t} \tag{23}$$

which essentially states that displacement over a duration of time t is the same as motion at the average speed over that same duration.

Aside: What about the "Big Five"?

Some teachers and textbooks will teach the "big five" kinematic equations, which also include

$$r(t) - r_0 = v(t)t - \frac{1}{2}at^2$$

In certain situations, it's more expedient to use this since it eliminates the initial velocity v_0, but a simple application of the first kinematic equation (7) to the second (11) recovers this equation almost immediately, so I find it unnecessary to elevate this equation to "official equation" status.

Solving 1D Kinematics Problems

Most 1D kinematics problems boil down to having a subset of $r(t) - r_0$, v_0, $v(t)$, a, and t and wanting to find another member. Note that the equations can be recast into just using $\Delta r(t) = r(t) - r_0$, so meaning that only displacements matter rather than absolute positions, so there are really only five variables, and each equation only uses four of them, so only three quantities are needed to figure the other two out.

Example: Drag Racing

A race car starts accelerating at a constant rate of 5 m/s^2 from rest. How long does it take for the car to travel 1 km?

Solution It doesn't look like we have a lot to go on here, but actually we do. We want to know how long it takes to travel 1 km, so that means we want to know what time t is required to make $r(t) - r_0 = 1$ km. The problem explicitly states the acceleration is 5 m/s^2, so we have two of three pieces of information. . . What is the third?

The key word here is "rest". Starting from rest means the race car starts with an initial velocity of 0, so we have $v_0 = 0$. Now we need only select an equation that has $r(t) - r_0$, v_0, a, and t in it. Looks like the second equation (11) will do! Let's solve for t, using the fact that $v_0 = 0$ to simplify things:

$$r(t) = r_0 + v_0 t + \frac{1}{2}at^2 \tag{24}$$

$$r(t) - r_0 = \frac{1}{2}at^2 \tag{25}$$

$$\frac{2\left[r(t) - r_0\right]}{a} = t^2 \tag{26}$$

$$\sqrt{\frac{2\left[r(t) - r_0\right]}{a}} = t \tag{27}$$

Before plugging numbers in, we should make sure the units make sense. Inside the square root, we have a distance $(r(t) - r_0)$ divided by an acceleration (a). Acceleration has units of distance per time squared, distance divded by acceleration has units of time squared, and the square root takes us back into plain old time. That gives us confidence we are at the right answer! Let's plug in numbers to get

$$\boxed{t = \sqrt{\frac{2\left[r(t) - r_0\right]}{a}} = \sqrt{\frac{2(1000\text{ m})}{5\text{ m/s}^2}} = 20\text{ s}} \tag{28}$$

Warning: Be Aware of Your Assumptions

These equations are only valid in the case of constant acceleration. When we study dynamics, we'll encounter situations where the acceleration an object experiences depends on its position, so acceleration will not be constant! Remember, the assumption we started with in our derivation was constant acceleration.

Extra Practice

Derivations

First, try to re-derive the kinematic equations yourself starting from the assumption of constant acceleration. Once you can do that and understand the derivation, you never need to memorize these equations! Also try deriving the fourth kinematic equation without using the first kinematic equation.

Problem Solving

Your textbook likely has many problems like the example. For more, check out the online resources at `wolftutoring.com`. Or try searching the internet for "kinematic equations problems".